

Dynamic Phase Transitions in Superconductivity

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In this Letter, the dynamic phase transitions of the time-dependent Ginzburg-Landau equations are analyzed using a newly developed dynamic transition theory and a new classification scheme of dynamics phase transitions. First, we demonstrate that there are two type of dynamic transitions, jump and continuous, dictated by the sign of a nondimensional parameter R . This parameter is computable, and depends on the material property, the applied field, and the geometry of domain that the sample occupies. Second, using the parameter R , precise analytical formulas for critical domain size, and for critical magnetic fields are derived.

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One central problem in the theory of superconductivity is the dynamical nature of the phase transition between a normal state, characterized by a complex order parameter ψ that vanishes identically, and a superconducting state, characterized by the order parameter that is not identically zero. In this Letter, we address this problem by conducting rigorous theoretical analysis on dynamic phase transitions for the time dependent Ginzburg-Landau (TDGL) model.

TDGL model. Let $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$) be a bounded domain, $\psi : \Omega \rightarrow \mathbb{C}$ the order parameter, $H : \Omega \rightarrow \mathbb{R}^n$ the magnetic field with A being the magnetic potential given by $H = \text{curl } A$, and H_a the applied field with potential A_a such that $\text{curl } A_a = H_a$. The nondimensional TDGL equations for $(\psi, \mathcal{A}, \phi)$, with \mathcal{A} being the deviation from the applied field, are given by:

$$\begin{aligned} \frac{\partial \psi}{\partial t} + i\phi\psi &= -(i\mu\nabla + A_a)^2\psi + \alpha\psi - 2A_a \cdot \mathcal{A}\psi \\ &\quad - 2i\mu\mathcal{A} \cdot \nabla\psi - |\mathcal{A}|^2\psi - \beta|\psi|^2\psi, \\ \zeta \left[\frac{\partial \mathcal{A}}{\partial t} + \mu\nabla\phi \right] &= -\text{curl}^2 \mathcal{A} - \gamma A_a |\psi|^2 \\ &\quad - \gamma \mathcal{A} |\psi|^2 - \frac{\gamma\mu}{2} i(\psi^* \nabla\psi - \psi \nabla\psi^*), \\ \text{div} \mathcal{A} &= 0. \end{aligned} \quad (1)$$

Here the nondimensional parameters are defined by

$$\begin{aligned} \alpha &= -2a\sqrt{b}m_s D/e_s^3 h, & \beta &= 2m_s D/h, \\ \mu &= hD/\sqrt{b}e_s, & \zeta &= 4\pi\sigma l e_s^2/c^2 h, \\ \gamma &= 4\pi e_s^2/m_s c^2 l, & \lambda &= \lambda(T) = (m_s c^2 b/4\pi e_s^2 |a|)^{1/2}, \\ \kappa &= \lambda/\xi, & \eta &= 4\pi\sigma D/c^2, \end{aligned}$$

where h is the Planck constant, e_s and m_s are the charge and mass of a Cooper pair, c is the speed of light, $\lambda = \lambda(T)$ is the penetration depth, $\xi(T)$ is the coherence length, τ for the relaxation time, and $\kappa = \lambda/\xi$ is the Ginzburg-Landau parameter.

The TDGL equations are supplemented with an initial condition for (ψ, \mathcal{A}) , a free-slip boundary condition for \mathcal{A} , and either the Neumann or the Dirichlet or the Robin boundary conditions for ψ ; see de Gennes [1]. Here We shall see in later discussions that the parameter α plays a key role in the phase transition of superconductivity, which is given in terms of dimensional quantities by [1]:

$$\alpha = \alpha(T) = \frac{2\sqrt{b}m_s D N_0}{e_s^3 h} \cdot \frac{T_c - T}{T_c}, \quad (2)$$

where N_0 the density of states at the Fermi level, and T_c the critical temperature where incipient superconductivity property can be observed.

Dynamic transition. As mentioned earlier, the study of dynamic phase transition problem for the TDGL equations is based on a new dynamic transition theory by the authors [2, 3]. The starting point of the theory is to put the TDGL equations into the perspective of an infinite-dimensional dynamical systems as follows:

$$\frac{du}{dt} = L_\lambda u + G(u, \lambda), \quad u(0) = u_0. \quad (3)$$

where $u : [0, \infty) \rightarrow H$ is the unknown function, H is a Hilbert space, L_λ is a linear operator, $G(u, \lambda)$ is a nonlinear operator, and λ is the system parameter. Then under proper physical conditions, the dynamical transitions of a basic state of (3) at a critical parameter λ_0 , where the principle of exchange of stabilities hold true, can be classified into three categories: continuous as shown in Figure 1, jump as shown in Figure 2, and mixed; see the above references for further details. This theory has been applied to the TDGL equations (1) to characterize the dynamic transition, will be used in the analysis hereafter in this Letter.

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Eigenvalue problem: Consider the following equation

$$(i\mu\nabla + A_a)^2\psi = \alpha\psi \quad \forall x \in \Omega, \quad (4)$$

with one of the boundary condition for ψ . There are an infinite real eigenvalue sequence of (4) as $0 \leq \alpha_1 < \alpha_2 < \dots < \alpha_k < \dots \rightarrow +\infty$. The eigenvalues of (4) always have even multiplicity. In this Letter, we consider only the case where the first eigenvalue α_1 of (4) has multiplicity two, i.e., α_1 is complex simple eigenvalue. We let

$$e = e_1 = \psi_{11} + i\psi_{12}, \quad e_2 = -\psi_{12} + i\psi_{11}. \quad (5)$$

be the eigenvectors corresponding to α_1 .

A necessary condition: In superconductivity, the parameter α in (1) can not exceed a maximal value $\alpha(T) \leq \alpha(0)$ because $T \geq 0$. Hence, a necessary condition for the possible phase transition from the normal state to superconducting states is:

$$\alpha_1 < \alpha(0) = \frac{2\sqrt{b}m_s D N_0}{e_s^3 h}, \quad (6)$$

where α_1 is the first eigenvalue of (4).

For the case where the Neumann boundary condition $\partial\psi/\partial n = 0$ on $\partial\Omega$, the first eigenvalue $\alpha_1 = 0$, which is independent of Ω , the geometry of the sample. Therefore, the condition (6) always holds true.

However for the Dirichlet and the Robin boundary conditions, the situations are different. We know that the first eigenvalue α_1 depends on Ω . We have for example $\lim_{|\Omega| \rightarrow 0} \alpha_1 = \infty$. Hence, the condition (6) implies that for the cases with either the Dirichlet or Robin boundary conditions, including the case where the sample is enclosed by a magnetic material or a normal metal, the volume of a sample must be greater than some critical value $|\Omega| > V_c > 0$. Otherwise no superconducting state occurs at any temperature.

Physical theory and experiments show that there is a critical applied magnetic field H_c by which a superconducting state will be destroyed, and H_c satisfies the following approximate equation near the critical temperature T_c which is given by (2)

$$H_c(T) = H_0(1 - T^2/T_c^2). \quad (7)$$

Equation (7) is an empirical formula. A related equation is the equation of critical parameter:

$$\alpha(T) = \alpha_1(H_a, \Omega), \quad (8)$$

where $\alpha(T)$ is the parameter in (1) and $\alpha_1 = \alpha_1(H_a, \Omega)$ the first eigenvalue of (4). It is expected that the applied magnetic field H_a satisfying (8) is the critical field H_c .

Definition of R : We start with the introduction of a crucial physical parameter, which completely determines the dynamical properties of the phase transition of the Ginzburg-Landau equations.

For the first eigenfunction e , the following Stokes problem has a unique solution:

$$\begin{aligned} \text{curl}^2 \mathcal{A}_0 + \nabla \phi &= |e|^2 A_a + \frac{\mu}{2} i(e^* \nabla e - e \nabla e^*), \\ \text{div} \mathcal{A}_0 &= 0, \\ \mathcal{A}_0 \cdot n|_{\partial\Omega} &= 0, \quad \text{curl} \mathcal{A}_0 \times n|_{\partial\Omega} = 0. \end{aligned} \quad (9)$$

Then we define a physical parameter R as follows

$$R = -\frac{\beta}{\gamma} + \frac{2 \int_{\Omega} |\text{curl} \mathcal{A}_0|^2 dx}{\int_{\Omega} |e|^4 dx}. \quad (10)$$

From (9) and (10) it is easy to see that the parameter R is independent of the choice of the first eigenvectors of (4) and $H_0 = \text{curl} \mathcal{A}_0$ given by (9) depend on the applied magnetic potential A_a and the geometric properties of Ω , the parameter R is essentially a function of A_a, Ω and physical parameters β, γ, μ .

As mentioned earlier, there are two phase transitions: Type-I and Type-II, determined by a simple parameter R defined by (10). This parameter R links the superconducting behavior with the geometry of the material, the applied field and the physical parameter's.

An equilibrium state $(\tilde{\psi}, \tilde{\mathcal{A}})$ of the TDGL equations (1) is called in the normal state if $\tilde{\psi} = 0$, and it is called in the superconducting state if $\tilde{\psi} \neq 0$. A solution (ψ, \mathcal{A}) of (1) is said in the normal state if (ψ, \mathcal{A}) is in a domain of attraction of a normal equilibrium state, otherwise (ψ, \mathcal{A}) is said in the superconducting state.

Dynamic phase transition for $R < 0$: By the phase transition theorems obtained by the authors [4], the critical temperature T_c^1 of superconducting transition satisfies $T_c^1 < T_c$ where T_c is as in (2) and $T = T_c^1$ satisfies (8). It is known that $\lim_{|A_a| \rightarrow \infty} \alpha_1(H_a, \Omega) = \infty$, which implies that the applied magnetic field H_a can not be very strong for superconductivity as required by condition (6).

We have shown in [4] that when $\alpha > \alpha_1$, the equations (1) bifurcate from $((\psi, \mathcal{A}), \alpha) = (0, \alpha_1)$ to a cycle of steady state solutions $(\psi_\alpha, \mathcal{A}_\alpha)$ which is an attractor attracting an open set $U \setminus \Gamma \subset H$. From the physical point of view, this theorem leads to the following properties of superconducting transitions in the case where $R < 0$:

FIRST, when the control temperature decreases (resp. increases) and crosses the critical temperature T_c^1 , there will be a phase transition of the sample from the normal to superconducting states (resp. from the superconducting to normal state).

SECOND, when the control temperature $T > T_c^1$, under a fluctuation deviating the normal state, the sample will be soon restored to the normal state. In addition, when $T < T_c^1$, under a fluctuation deviating both the normal and superconducting states, the sample will be soon restored to the superconducting states.

THIRD, in general, the supercurrent given by

$$J_s(\alpha) = -\gamma(A_a + \mathcal{A}_\alpha)|\psi_\alpha|^2 - \frac{\gamma\mu}{2}i(\psi_\alpha^* \nabla \psi_\alpha - \psi_\alpha \nabla \psi_\alpha^*)$$

is nonzero, i.e., $J_s \neq 0$ for $T < T_c^1$ ($\alpha_1 < \alpha$).

FOURTH, the order parameter ψ_α and supercurrent $J_s(\alpha)$ depend continuously on the control temperature T (or the parameter α):

$$\psi_\alpha \rightarrow 0, \quad J_s(\alpha) \rightarrow 0 \quad \text{if } T \rightarrow (T_c^1)^- \text{ (or } \alpha \rightarrow \alpha_1^+).$$

FIFTH, the superconducting state of the system is dominated by the lowest energy eigenfunction of (4) in the sense that $\psi = C|\alpha - \alpha_1|^{1/2}e + o(|\alpha - \alpha_1|^{1/2})$.

SIXTH, the phase transition is of second order in the Ehrenfest sense with the critical exponent $\beta = 1/2$, and its phase diagram is as shown in Figure 1:

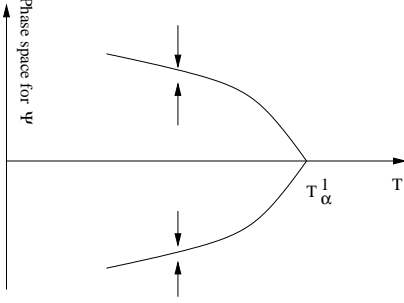


FIG. 1: The continuous transition with $R < 0$.

Dynamic phase transitions with $R > 0$: Consider a material described by the TDGL model with $R > 0$. There are two transition temperatures T_c^0 and T_c^1 ($T_c^0 > T_c^1$) such that

$$\alpha(T_c^i) = \alpha_i \quad (i = 0, 1) \quad \text{with} \quad \alpha_0 < \alpha_1,$$

and the following phase transition properties hold true:

FIRST, when the control temperature T decreases and crosses T_c^1 or equivalently α increases and crosses α_1 , the stability of the normal state changes from stable to unstable.

SECOND, when $T_c^1 < T < T_c^0$ ($\alpha_0 < \alpha < \alpha_1$), physically observable states consist of the normal state and the superconducting states in Σ_α^2 (see Figure 8.20). When $T < T_c^1$ ($\alpha_1 < \alpha$), physically observable states are in Σ_α^2 .

THIRD, when the control temperature T is in the interval $T_c^1 < T < T_c^0$ (or $\alpha_0 < \alpha < \alpha_1$), the superconducting states in Σ_1^α are unstable, i.e., with a fluctuation deviating a superconducting state in Σ_1^α , transition to either the normal state or a superconducting state in Σ_2^α will occur.

FOURTH, at the critical temperature T_c^0 (resp. at T_c^1) of the phase transitions, there is a jump from the superconducting states to the normal state (resp. from the normal state to superconducting states).

FIFTH, the other energy-level eigenfunctions possibly have a stronger influence for the superconducting states.

SIXTH, in the temperature interval $T_c^1 < T < T_c^0$, phase transitions occur and are accompanied with the latent heat to appear.

Based on the conclusions (1)-(4) above, we can draw the phase diagram in Figure 2, where the critical temperature T_c in the interval $T_c^1 < T_c < T_c^0$ is the transforming point.

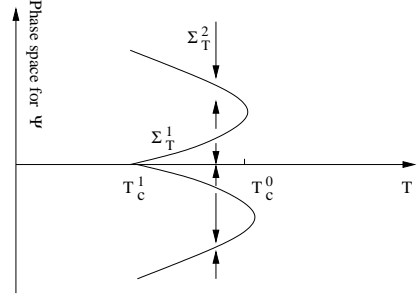


FIG. 2: The jump transition with $R > 0$.

Critical Sample Size and Critical Magnetic Fields: Theories and experiments illustrate that the geometry of samples Ω and applied magnetic fields H_a have important influences for the superconducting behaviors. In the following, we shall apply the formula (10) and eigenvalue equation (4) to discuss this problem.

For $0 < L < \infty$ and $h > 0$, let $\Omega_0 = D_0 \times (0, h) \subset \mathbb{R}^3$, $x' = (x_1, x_2) \in D_0$, $0 \in D_0$, and

$$\Omega(L) = \{(Lx', x_3) \mid x' \in D_0, 0 < x_3 < h\}.$$

Let the applied field be given by $H_a = H_a(x') = (0, 0, H(x'))$, where $x' = (x_1, x_2) \in D_0$. Then H_a induces an applied field \tilde{H}_a on $\Omega(L)$ by $\tilde{H}(y) = H(y/L)$ for any $y = Lx'$ with $x' \in D_0$.

Let $H_a = \text{curl} A_a$, i.e., $H = \frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1}$, where

$$A_a = (A_1(x'), A_2(x'), 0). \quad (11)$$

Then we can show that the nondimensional parameter $R = R(L, H)$ on $\Omega(L)$ is given by

$$R(L, H) = -\kappa^2 \mu^2 + 2L(p_3 L^4 + p_2 L^2 + p_1), \quad (12)$$

where for any $0 < L < \infty$,

$$\begin{cases} p_3 = \frac{\int_{D_0} |\text{curl} A_0|^2 dx'}{\int_{D_0} |e|^4 dx'} > \delta > 0, \\ \text{curl}^2 A_0 + \nabla \phi = |e|^2 A_a \quad \text{in } x' \in D_0, \\ p_1 = \frac{4\mu^2 \int_{D_0} |\text{curl} B_0|^2 dx'}{\int_{D_0} |e|^4 dx'}, \\ \text{curl}^2 B_0 + \nabla \phi = e_{12} \nabla e_{11} \quad \text{in } x' \in D_0, \\ p_2 = \frac{4\mu \int_{D_0} \text{curl} A_0 \cdot \text{curl} B_0 dx'}{\int_{D_0} |e|^4 dx'}. \end{cases}$$

Here $\delta > 0$ is independent of L , and $e(x') = e_{11} + ie_{12}$ is the first eigenfunction of

$$\begin{aligned} (i\mu \nabla + L^2 A_a)^2 e &= L^2 \alpha e \quad \text{in } D_0, \\ \frac{\partial e}{\partial n} &= 0 \quad \text{on } \partial D_0. \end{aligned} \quad (13)$$

Formula (12) has many applications. In particular, we derive the following effort of the domain and the applied

field. It is easy to observe that p_3 is essentially a $|H|^2$ term, p_2 is a $|H|$, and p_1 is a zeroth order of $|H|$.

First, by (12), we derive that given a superconducting material, and an applied field $H_a \neq 0$, there is a critical scale $L_0 > 0$ such that the phase transition in $\Omega(L)$ is a continuous transition if $L < L_0$, and is a jump transition if $L > L_0$. In addition, L_0 is the unique real root of

$$\alpha_3 L_0^5 + \alpha_2 L_0^3 + \alpha_1 L_0 = \frac{1}{2} \kappa^2 \mu^2.$$

This theoretical conclusion is known experimentally for the small sample $\Omega(L)$ (i.e., $1 \gg L$) case.

Second, for Type I superconducting materials ($\kappa^2 < 1/2$), there are three critical magnetic fields $H_{c1} < H_{c2} < H_{c3}$ such that the following hold true:

1. If $0 < H < H_{c1}$, then the transition is Type-I and the superconducting state is in the Meissner state;
2. If $H_{c1} < H < H_{c2}$, then transition is Type-II and in the Meissner state,
3. If $\alpha_1(A_{c2}) = \alpha(0)$, then for any $H_{c2} < H$, it is in the normal state.
4. If $\alpha_1(A_{c2}) < \alpha(0)$, then
 - (a) if $H_{c2} < H < H_{c3}$, then transition is Type-II and in the mixed state, and
 - (b) if $H_{c3} < H$, then it is in the normal state and $\alpha_1(A_{c3}) = \alpha(0)$.

Third, for type II superconductors, the results are similar:

1. If $0 < H < H_{c1}$, then the transition is Type-I and in the Meissner state,

2. If $H_{c1} < H < H_{c2}$, then transition is Type-I and in the mixed state,
3. If $\alpha_1(A_{c2}) = \alpha(0)$, then for any $H_{c2} < H$, it is in the normal state.
4. If $\alpha_1(A_{c2}) < \alpha(0)$, then
 - (a) if $H_{c2} < H < H_{c3}$, then transition is Type-II and in the mixed state, and
 - (b) if $H_{c3} < H$, then it is in the normal state and $\alpha_1(A_{c3}) = \alpha(0)$.

We remark that the properties above are well known by the Abrikosov theory and physical experiments. However, here conclusions are derived rigorously using the recently developed dynamic transition theory, and the critical fields can be solved from (12).

Fourth, for a small sample, there is a remarkable difference that the first eigenvalue $\alpha_1(A_a)$ of with the Neumann boundary condition is bigger for small applied magnetic field $H_a = \text{curl} A_a$. Hence the critical magnetic field $H_c = \text{curl} A_c$ satisfying that $\alpha_1(A_c) = \alpha(0)$ is very small, which implies that for a small sample, the superconductivity is only in the Meissner state.

Finally we would like to mention that there have been extensive studies on bifurcation and stability analysis for superconductivity; see among others [5, 6, 7, 8, 9]. The study in this article is based on a newly developed dynamic transition theory by the authors [2, 3]. With this new theory, many long standing problems in phase transition problems in science and engineering are becoming more accessible. In particular, applications are made for various models from science and engineering, including, in particular, problems in statistical physics, classical and geophysical fluid dynamics.

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